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APPLICATION OF HANKEL TRANSFORMS TO THE SOLUTION OF AXISYMMETRIC PROBLEMS WHEN THE MODULUS OF ELASTICITY IS A POWER FUNCTION OF DEPTH

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A. E. PURO
(Tallin)

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The application of Hankel transforms to the three-dimensional axisymmetric problems of the theory of elasticity, in the case when the modulus of elasticity is a power function of depth, leads to a system of ordinary differential equations [1] whose solution presents some mathematical difficulty. Therefore, in [1, 2] the solution of these problems has been carried out by applying transformations expounded in [3].

In the sequel we construct the fundamental system of solutions of the ordinary differential equations mentioned above and we give the solution for two boundary value problems in the case of very special conditions.

1. In the case of axial symmetry, the displacement equations of the theory of elasticity have the form

$$(\lambda+2\mu)\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right]+(\lambda+\mu)\frac{\partial^2 w}{\partial r\partial z}+\mu\frac{\partial^2 u}{\partial z^2}+\frac{\partial\mu}{\partial z}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right]=0$$

$$\mu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right]+(\lambda+\mu)\left[\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right]+(\lambda+2\mu)\frac{\partial^2 w}{\partial z^2}-\frac{\partial\lambda}{\partial z}\left[\frac{\partial u}{\partial z}+\frac{u}{r}\right]+\frac{\partial\,(2\mu+1)}{\partial z}\frac{\partial w}{\partial z}=0$$

We apply the Fourier method, by using the Hankel's transforms in the following form

$$u(r,z)=\int_0^\infty \varphi(s,z)\,J_1(sr)\,ds,\quad w(r,z)=\int_0^\infty f(s,z)\,J_0(sr)\,ds$$

We obtain a system of ordinary differential equations with variable coefficients [1]

$$\varphi'' = -\frac{\mu'}{\mu}\varphi' + \frac{\lambda+\mu}{\mu} sf' + \frac{\lambda+2\mu}{\mu} s^2\varphi + \frac{\mu'}{\mu} sf \tag{1.1}$$

$$f'' = -\frac{\lambda+\mu}{\lambda+2\mu} s\varphi' - \frac{\lambda'+2\mu'}{\lambda+2\mu} f' - \frac{\lambda'}{\lambda+2\mu} s\varphi + \frac{\mu s^2}{\lambda+2\mu} f$$

Let us consider the case when Lamé coefficients have the form $\mu = \mu_0 z^\beta$, $\lambda = \lambda_0 z^\beta$. We perform the change: $k = zs$, $g = (\lambda_0 + 2\mu_0) / \mu_0$ and we try to represent the system of differential equations as follows:

$$\left[E\frac{d}{dk} - A\right]\left[E\frac{d}{dk} - B\right]\left[\begin{array}{c}\varphi\\ f\end{array}\right] = 0$$

Here A, B are the coefficient matrices and E is the identity matrix. The fundamental system of solutions can be represented in this case in the form

$$\psi_b,\ \psi_b\int_{k_0}^{k} \psi_b^{-1}\psi_a dk$$

where ψ_b and ψ_a is the fundamental system of solutions of the equations

$$\left[E\frac{d}{dk} - B\right]\left[\begin{array}{c}\varphi\\ f\end{array}\right] = 0 \tag{1.2}$$

$$\left[E\frac{d}{dk} - A\right]\left[\begin{array}{c}\varphi\\ f\end{array}\right] = 0 \tag{1.3}$$

We write the initial system (1.1) in matrix form

$$\left[E\frac{d^2}{dk^2} - G\frac{d}{dk} - H\right]\left[\begin{array}{c}\varphi\\ f\end{array}\right] = 0$$

$$G = \left\|\begin{array}{cc} -\dfrac{\beta}{k} & g-1 \\ -\dfrac{g-1}{g} & -\dfrac{\beta}{k}\end{array}\right\|, \qquad H = \left\|\begin{array}{cc} g & \dfrac{\beta}{k} \\ -\dfrac{g-2}{g}\dfrac{\beta}{k} & \dfrac{1}{g}\end{array}\right\|$$

For the determination of the unknown matrices A and B we obtain a system of matrix equations

$$A + B = G, \qquad \frac{d}{dk}B - AB = H$$

We seek the solution of the matrix system in the form of the series

$$A = \sum_n A_n k^{-n}, \qquad B = \sum_n B_n k^{-n} \tag{1.4}$$

As a result of such a substitution we obtain a system of matrix algebraic equations.

Let us consider the case when the series (1.4) are truncated and consist of two terms, i.e.

$$A = A_0 = A_1 k^{-1}, \quad B = B_0 + B_1 k^{-1}$$

In this case we obtain the following algebraic system of matrix equations:

$$A_0 + B_0 = G_0, \ \ A_0 B_0 = -H_0, \ \ A_1 + B_1 = G_1 \tag{1.5}$$

$$A_0 B_1 + A_1 B_0 = -H_1, \ \ A_1 B_1 + B_1 = 0$$

Here

$$G_0=\left\|\begin{matrix}0 & (g-1)\\ -\frac{g-1}{g} & 0\end{matrix}\right\|,\qquad G_1=\left\|\begin{matrix}-\beta & 0\\ 0 & -\beta\end{matrix}\right\|$$

$$H_0=\left\|\begin{matrix}g & 0\\ 0 & \frac{1}{g}\end{matrix}\right\|,\qquad H_1=\left\|\begin{matrix}0 & \beta\\ -\frac{g-2}{g}\beta & 0\end{matrix}\right\|$$

The system (1.5) does not always have a solution, in particular, it has a solution when the eigenvalues of the matrices A_0 and B_0 coincide. The system of the first two matrix equations (1.5) is equivalent to the system

$$A_0G_0 - A_0^2 + H_0 = 0,\quad G_0B_0 - B_0^2 + H_0 = 0$$

The solution of these equations is given in [4]

$$A_0 = [-H_0 - m_3m_4E]\,[G_0 - (m_3 + m_4)E]^{-1}$$
$$B_0 = [G_0 - (m_1 + m_2)\,E]^{-1}\,[-H_0 - m_1m_2E]$$

where the eigenvalues of the matrices A_0 and B_0 are obtained from the equation

$$\det\,[G_0m - m^2E + H_0] = 0$$

The system formed by the third and the fourth equations of (1.5) can be reduced to the equation

$$A_0B_1 - B_1B_0 = -H_1 - G_1B_0$$

If the eigenvalues of the matrices A_0 and B_0 are distinct, the system has a unique solution, if these eigenvalues are identical, then the system is either inconsistent or it has infinite number of solutions in the form of the sum of a particular solution and the general solution of the homogeneous equation. The solution of the system formed by the third and the fifth equations of (1.5), in the case when G_1 is a multiple of the identity matrix $(G_1 = -\beta E)$, has the form

$$A_1 = -\beta E - B_1,\qquad B_1 - T^{-1}\left\|\begin{matrix}0 & 0\\ 0 & 1-\beta\end{matrix}\right\|T\qquad \text{(solution 1)}$$

$$A_1 = -E,\quad B_1 = (1-\beta)\,E\quad \text{(solution 2)}$$

Here T is any nonsingular matrix.

As a result we obtain the following solutions of Eqs. (1.5):

Solution 1

$$A_0=\left\|\begin{matrix}0 & g\\ g^{-1} & 0\end{matrix}\right\|,\qquad B_0=\left\|\begin{matrix}0 & -1\\ -1 & 0\end{matrix}\right\|$$

$$A_1=\frac{1}{1-g}\left\|\begin{matrix}(\beta+g) & -\sqrt{l_1l_2g}\\ \sqrt{l_1l_2g^{-1}} & \beta g-2\beta-1\end{matrix}\right\|$$

$$B_1=\frac{1}{1-g}\left\|\begin{matrix}-l_1 & \sqrt{l_1l_2g}\\ \sqrt{l_1l_2g^{-1}} & l_2\end{matrix}\right\|,\qquad l_1=\beta(2-g)+g,\quad l_2=1+\beta$$

Solution 2

$$A_0=\left\|\begin{matrix}0 & g\\ g^{-1} & 0\end{matrix}\right\|,\qquad B_0=\left\|\begin{matrix}0 & -1\\ -1 & 0\end{matrix}\right\|,\qquad A_1=-E,\qquad B_1=-\frac{2}{g-1}E$$

$$\beta = g + 1/g - 1$$

2. We consider both solutions.

Solution 1. The system of equations of the first degree (1.2),(1.3) can be reduced to the Whittaker's equation with solutions in the form of Whittaker's functions (2.1) and (2.2) corresponding to $l = {}^1/_2\sqrt{l_1 l_2 g^{-1}}$

$$W_{-l,\,\beta/2}(2k), \qquad W_{l,\,\beta/2}(2k) \tag{2.1}$$

$$W_{l,\,\beta/2}(-2k), \quad W_{-l,\,\beta/2}(2k) \tag{2.2}$$

In certain special cases, these solutions can be expressed as polynomials [5].

For $\beta = g / (g-2)$, the solution can be expressed in terms of Bessel's functions and the fundamental system of solutions is obtained from the expressions

$$\psi_b = k^{g-4/2(g-2)} \begin{Vmatrix} K_{\beta/2-1}(k) & -I_{\beta/2-1}(k) \\ K_{\beta/2}(k) & I_{\beta/2}(k) \end{Vmatrix}$$

$$\psi_a = k^{-\beta/2} \begin{Vmatrix} -gK_{-\beta/2}(k) & gI_{-\beta/2}(k) \\ K_{1-\beta/2}(k) & I_{1-\beta/2}(k) \end{Vmatrix}$$

For $\beta = -1$, the fundamental system is determined from the expressions

$$\psi_b = \begin{Vmatrix} ke^{-k} & ke^{k} \\ (1+k)e^{-k} & (1-k)e^{k} \end{Vmatrix} \tag{2.3}$$

$$\psi_a = \frac{1}{k} \begin{Vmatrix} -ge^{k}(1-k) & -ge^{-k}(1+k) \\ ke^{k} & ke^{-k} \end{Vmatrix}$$

Solution 2. In this case the condition $\beta = (g+1)/(g-1)$ is imposed on β. The fundamental system is determined from the expressions

$$\psi_b = k^{-2/g-1} \begin{Vmatrix} e^{k} & e^{-k} \\ -e^{k} & e^{-k} \end{Vmatrix}, \qquad \psi_a = \frac{1}{k} \begin{Vmatrix} e^{k} & e^{-k} \\ \dfrac{e^{k}}{g} & -\dfrac{e^{-k}}{g} \end{Vmatrix} \tag{2.4}$$

Example 1. In geophysics it presents interest to solve the Boussinesq problem for $g = 2.8$, which corresponds to

$$\lambda_0 = 2.4 \times 10^{11} \text{ g/cm sec}^2, \quad \mu_0 = 3 \times 10^{11} \text{ g/cm sec}^2$$

For the sake of simplicity we consider the case when $g = 3$, $\beta = 2$, which corresponds to the Solution 2. The boundary conditions have the form

$$\tau_{rz}\Big|_{z=z_0} = 0, \quad \sigma_z\Big|_{z=z_0} = -\delta(r), \quad \lim u(r,z)\Big|_{z\to\infty} = 0, \qquad \lim w(r,z)\Big|_{z\to\infty} = 0$$

The fundamental system of solutions is

$$\begin{Vmatrix} u(sz) \\ w(sz) \end{Vmatrix} = \frac{1}{zs} \begin{Vmatrix} P(s)e^{sz} + Q(s)e^{-sz} + R(s)e^{zs}(zs+1) + U(s)e^{-zs}(zs-1) \\ -P(s)e^{sz} + Q(s)P^{-sz} + R(s)e^{zs}(zs-1) + U(s)e^{-zs}(zs+1) \end{Vmatrix}$$

From the conditions at infinity we obtain $P(s) \equiv R(s) \equiv 0$. Substituting the solution into the boundary conditions, we obtain the values of $Q(s)$, $U(s)$.

Finally, the solutions have the form

$$u(r,z)=\frac{1}{2\pi\mu_0 z_0^2}\int_0^\infty \frac{(z_0 s)^2\,[2sz_0(z-z_0)+s(z-2z_0)]\,e^{-(z-z_0)s}}{zs\,[4(z_0 s)^2+12z_0 s+6]}J_1(rs)\,ds$$

$$w(r,z)=\frac{1}{2\pi\mu_0 z_0^2}\int_0^\infty \frac{(z_0 s)^2\,[2sz_0(z-z_0)+s(z+2z_0)+2]\,e^{-(z-z_0)s}}{zs\,[4(z_0 s)^2+12z_0 s+6]}J_0(rs)\,ds$$

Example 2. A thick plate under the action of a concentrated force. We take Lamé's coefficients in the form $\lambda=\lambda_0/z$, $\mu=\mu_0/z$, which corresponds to the Solution 1. For $z=0$ the plate is absolutely rigid.

The boundary conditions are

$$\tau_{rz}\Big|_{z=z_0}=0,\quad \sigma_z\Big|_{z=z_0}=\delta(r),\quad u(r,z)\Big|_{z=0}=0,\quad w(r,z)\Big|_{z=0}=0$$

Making use of the fundamental system of solutions (2.3), we obtain the solution at the boundary $z=z_0$ in the form

$$u(r)=\frac{1}{2\pi\mu_0 z_0}\int_0^\infty \frac{g\,[z_0 s(1+\mathrm{ch}(2z_0 s))-\mathrm{sh}(2z_0 s)]}{2z_0 s\,[z_0 s(g-1)\chi+2g\,\mathrm{sh}(2z_0 s)+2z_0 s]}J_1(rs)\,ds$$

$$\chi=\int_0^{2z_0 s}\frac{\mathrm{ch}(k)-1}{k}\,dk$$

The formula for $w(r)$ has a similar form.

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